Supplemental material for pump-controlled directional light emission from random lasers

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PUMP PROFILE BASIS

To first specify and then optimize the pump profile for directional emission we expand the two-dimensional pump profile $F(\mathbf{x})$ on the disk-shaped random laser of radius R using a Bessel function basis of the form $J_m(j_{m,n}|\mathbf{x}|/R)\exp[-im\varphi]$. Here the $j_{m,n}$ are the Bessel function roots, φ the polar angle, and R the radius of the disk within which the scatterers are randomly distributed. This basis has the suitable property of vanishing at the disk boundary outside of which no pump is applied. Inside the disk we expand the real and strictly positive pump function $F(\mathbf{x})$ as follows,

$$F(\mathbf{x};\beta_{m,n}) = \left|\sum_{m,n}\beta_{m,n}J_m(j_{m,n}\frac{|\mathbf{x}|}{R})\exp[-im\varphi]\right|^2, \tag{1}$$

where $\beta_{m,n}$ are complex expansion coefficients. In our numerical calculations we restrict the summation in Eq. (1) to the limits $m \in [-m_{\max}, m_{\max}]$ and $n \in [0, n_{\max}]$, corresponding to a finite resolution of the pump beam in radial and azimuthal direction, respectively. The choice which we make for the $(2m_{\max}+1)\times(n_{\max}+1)$ complex coefficients $\beta_{m,n}$ will determine the threshold laser mode $\bar{\Psi}_1(\mathbf{x})$. In the calculations presented in the main part of the article (see Fig. 3), we choose $n_{\max} = 8$ and $m_{\max} = 8$. Note that, although not explicitly shown here, one can reduce the complex parameters $\beta_{m,n}$ to purely real ones since only the absolute square of the superposition of states is considered.

EMISSION FREQUENCY BEHAVIOR

In the course of optimizing the directionality of the first TLM the changes in the pump profile also lead to slight variations of the laser frequency around the peak gain frequency of

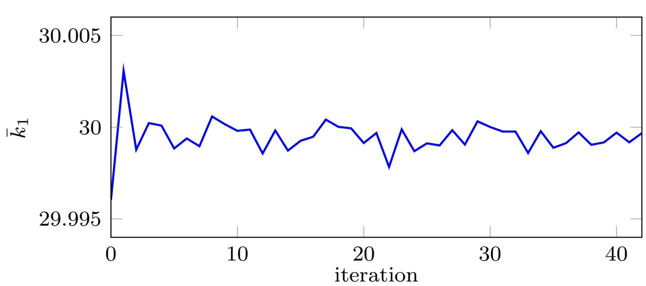

FIG. S1. Variation of the emission frequency $\bar{k}_1$ of the first threshold laser mode in the course of the optimization procedure (the same parameters were used here as for Fig. 3d in the main text).

$30\,\mu m^{-1}$. For the specific case displayed in Fig. 3d of the main text this frequency variation takes on the form shown in Fig. S1. Note that for the first few iteration steps this variation is larger since the pump profile is modified more strongly here as compared to later steps of the optimization procedure.

DERIVATION OF THE GRADIENT OF THE DIRECTIONALITY

In order to efficiently calculate the gradient of the directionality $\mathcal{D}$ with respect to the complex coefficients $\{\beta_{m,n}\}$ we approximate this gradient with the help of the TCF states which are defined as follows

$$\left\{\nabla^2 + \bar{k}_1^2\left[\varepsilon_c(\mathbf{x}) + \eta_n(\bar{k}_1)F(\mathbf{x},\bar{k}_1)\right]\right\} u_n(\mathbf{x}) = 0\,, \tag{2}$$

and which satisfy the self-orthogonality relation

$$\int_{r<R} u_i(\mathbf{x})F(\mathbf{x})u_j(\mathbf{x})\mathrm{d}\mathbf{x} = \delta_{i,j}. \tag{3}$$

The basis itself is calculated at the frequency $\bar{k}_1$ of the first threshold laser mode $\bar{\Psi}_1$, for which we optimize the angular emission pattern to match the targeted emission pattern $G(\varphi)$. In order to reduce the terminology in the following we rewrite, without loss of generality, the set of complex coefficients $\{\beta_{m,n}\}$ as a set of twice as many real coefficients α_i, where the imaginary part of the coefficients $\{\beta_{m,n}\}$ is moved into the corresponding basis function $f_i(\mathbf{x})$, such that,

$$F(\mathbf{x};\beta_{m,n}) \;=\; \left|\sum_{m,n}\beta_{m,n}J_m(j_{m,n}\frac{|\mathbf{x}|}{R})\exp[-im\varphi]\right|^2 \tag{4}$$

$$=: \left|\sum_i \alpha_i f_i(\mathbf{x})\right|^2. \tag{5}$$

We will now derive the gradient of the directionality $\mathcal{D}$ with respect to these coefficients α_i. The first part of this derivation follows directy from differentiation of $\mathcal{D}$ with respect to

α_j,

$$\frac{\partial}{\partial\alpha_j}\mathcal{D}[\{\alpha_i\}] = \partial_{\alpha_j}\int \tilde{G}(\varphi)\frac{\mathrm{FFP}(\{\alpha_i\},\varphi)}{\sqrt{\int \mathrm{FFP}^2(\{\alpha_i\},\varphi')\mathrm{d}\varphi'}}\mathrm{d}\varphi \tag{6}$$

$$= \int \tilde{G}(\varphi)\Bigg[\frac{\partial_{\alpha_j}\mathrm{FFP}(\{\alpha_i\},\varphi)}{\sqrt{\int \mathrm{FFP}^2(\{\alpha_i\},\varphi')\mathrm{d}\varphi'}} - \tag{7}$$

$$- \frac{\mathrm{FFP}(\{\alpha_i\},\varphi)\int \mathrm{FFP}(\{\alpha_i\},\varphi')\partial_{\alpha_j}\mathrm{FFP}(\{\alpha_i\},\varphi')\mathrm{d}\varphi'}{\left(\int \mathrm{FFP}^2(\{\alpha_i\},\varphi')\mathrm{d}\varphi'\right)^{\frac{3}{2}}}\Bigg]\mathrm{d}\varphi, \tag{8}$$

where $\tilde{G}(\varphi) = G(\varphi)/\sqrt{\int G(\varphi')^2\mathrm{d}\varphi'}$ is the normalized reference emission profile and FFP is the angular far-field emission profile of the first TLM. The latter is determined as the real part of the complex Poynting vector at $r \to \infty$ which can be simplified to

$$\mathrm{FFP}(\varphi) := \lim_{r\to\infty}\frac{r}{2\bar{k}_1}\mathrm{Im}[\bar{\Psi}_1^*(r,\varphi)\partial_r\bar{\Psi}_1(r,\varphi)]\,. \tag{9}$$

The partial derivative of FFP with respect to the coefficients α_j is given by

$$\partial_{\alpha_j}\mathrm{FFP}(\{\alpha_i\},\varphi) = \lim_{r\to\infty}\frac{r}{2\bar{k}_1}\mathrm{Im}\left[(\partial_{\alpha_j}\bar{\Psi}_1^*)\partial_r\bar{\Psi}_1 + \bar{\Psi}_1^*\partial_r(\partial_{\alpha_j}\bar{\Psi}_1)\right]. \tag{10}$$

For reasons of improved efficiency we express the gradient of the TLM $\partial_{\alpha_j}\bar{\Psi}_1$ in terms of the TCF states u_n. Note, that the first TLM $\bar{\Psi}_1$ is equivalent to a TCF state $u_{\bar{n}}$ evaluated at the threshold laser frequency $\bar{k}_1$, i.e., $\bar{\Psi}_1(\mathbf{x}) = u_{\bar{n}}(\mathbf{x},\bar{k}_1)$. With the help of this equivalence the gradient of $\bar{\Psi}_1$ is formally defined as

$$\partial_{\alpha_j}u_{\bar{n}} = \lim_{h\to 0}\frac{u_{\bar{n}}(\{\alpha_i + h\delta_{i,j}\}) - u_{\bar{n}}(\{\alpha_i\})}{h}. \tag{11}$$

We can now proceed and use perturbation theory in order to derive an expression for $\partial_{\alpha_j}u_{\bar{n}}$ explicitly. With the operator $\hat{L} = -\nabla^2 - \bar{k}_1^2\varepsilon_c$ we can rewrite the original eigenvalue problem Eq. (2) as

$$\hat{L}u_n = \eta_n\bar{k}_1^2F(\{\alpha_i\})u_n \tag{12}$$

and the "perturbed" eigenvalue problem as

$$\hat{L}\tilde{u}_n = \tilde{\eta}_n\bar{k}_1^2F(\{\alpha_i\ + h\delta_{i,j}\})\tilde{u}_n, \tag{13}$$

where the perturbed state $\tilde{u}_n = u_n(\{\alpha_i + h\delta_{i,j}\})$ and the perturbed eigenvalue $\tilde{\eta}_n = \eta_n(\{\alpha_i +$

$h\delta_{i,j}\}$). The perturbed pump profile can be approximated by

$$F(\{\alpha_i + h\delta_{i,j}\}) = \left|\sum_i \alpha_i f_i(x) + h f_j(x)\right|^2 \tag{14}$$

$$= F(\{\alpha_i\}) + h \underbrace{\sum_i 2\alpha_i \mathrm{Re}(f_i f_j^*)}_{F'(\{\alpha_i\})} + \mathcal{O}(h^2). \tag{15}$$

In a next step we insert this approximation, together with the following ansatz for the perturbed eigenstate,

$$\tilde{u}_{\bar{n}} \approx u_{\bar{n}} + h \sum_{\substack{i=1 \\ i \neq \bar{n}}}^{N} c_{i,\bar{n}} u_i, \tag{16}$$

and for the eigenvalue, $\tilde{\eta}_n = \eta_n + h\eta'_n$, into Eq. (13) ($N$ denotes here the size of the TCF basis). By discarding terms of $\mathcal{O}(h^2)$ and making use of the self-orthogonality of the TCF-states [Eq. (3)] we obtain the expansion coefficients of the perturbed TCF state,

$$c_{i,\bar{n}} = \frac{\eta_{\bar{n}}}{\eta_i - \eta_{\bar{n}}} \int_{r<R} u_i(\mathbf{x}) F'(\{\alpha_i\}) u_{\bar{n}}(\mathbf{x}) \mathrm{d}\mathbf{x}, \tag{17}$$

where R is the radius of our random laser disk. This expression can then be inserted back into Eq. (16), which yields

$$\partial_{\alpha_j} u_{\bar{n}} = \sum_{\substack{i=1 \\ i \neq \bar{n}}}^{N} c_{i,\bar{n}} u_i. \tag{18}$$

With this expression and Eq. (10), the desired approximate gradient of the directionality measure $\mathcal{D}$ with respect to the coefficients α_i, Eq. (8), is now easily evaluated.