

Supplementary Material for Perfectly absorbing exceptional points and chiral absorbers

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S1. EP PARAMETER VALUES

Here we state the parameter values used for the calculations reported in each of the figures. Bold type indicates parameters that were used in tuning to the EP.

TABLE I. EP Parameters for Fig 1 (bold indicates fine-tuned)

	Fig. a-c	Fig. d-f	Fig. g-i
Grating high index	2	2	2
Grating low index	1.5	1.5	1.5
L_1	1.2500	1.2560	1.2566
L_2	1.4	1.2566	1.2566
n'_1	2	2	1.9981
in''_1	0.0382i	0.0043i	0.0037i
n'_2	2	2	2
in''_2	0.0192i	0.0472i	0.0609i
EP frequency ω_0	5.0199	5.0012	5.0022

TABLE II. EP Parameters for Fig 2 (bold indicates fine-tuned)

Index of disk	1.5+ 0.0021i
Radius of disk	1
Index of scatterers	1.5
Radius of scatterers	0.05
Distance of scatter 1	0.04
Distance of scatter 2	0.0454
Angle between scatterers	156.30°
EP frequency ω_0	15.126

S2. SELF-ORTHOGONALITY AND INTEGRAL RELATION FOR HELMHOLTZ EP

In this section we discuss biorthogonality and self-orthogonal EPs in open problems, and derive a previously-noted [1–3] relation between the unconjugated inner product of the wave operator eigenfunctions and the unconjugated “norm” of the S -matrix eigenvectors. For simplicity, we focus on the scalar Helmholtz operator in one dimension ($c = 1$, $\omega = k$), over a domain of length

TABLE III. EP Parameters for Fig 3 (bold indicates fine-tuned)

Disk index	2
WGM mode number q	15
Grating real index high	2.0149
Grating imag index high	0.0153
Grating width ϕ	2°
Offset angle χ	8.9400°
Grating periodicity P	30 = 2 q
Disk radius	1
Waveguide width	0.08
Waveguide distance	0.16
EP frequency ω_0	9.3230

2 L , and with purely incoming boundary conditions (appropriate for CPA):

$$\{\nabla^2 + \varepsilon(x)\omega_m^2\} \psi_m = 0 \quad [\nabla\psi_m = \mp i\omega_m\psi_m]_{\pm L}. \quad (1)$$

The equivalent closed system, say with Dirichlet conditions $\psi(\pm L) = 0$, defines an ordinary eigenvalue problem $\hat{A}\psi_m = -k_m^2\psi_m$, where $\hat{A} \equiv \varepsilon^{-1}\nabla^2$. If we define the inner product as

$$(\psi, \phi) \equiv \int_{-L}^L dx \psi \varepsilon \phi, \quad (2)$$

then \hat{A} is symmetric, in the sense that $(\psi, \hat{A}\phi) = (\hat{A}\psi, \phi)$ for any ψ, ϕ satisfying the boundary conditions. The biorthogonality relation between eigenfunctions follows, $(\psi_m, \psi_n) \propto \delta_{mn}$, which guarantees that in a *closed* system EPs are self-orthogonal. On the other hand, the CPA problem is not an eigenvalue problem in the usual way, as the boundary conditions depend on the frequency ω , so that the problem is defined self-consistently in its eigenvalue. Consequently $(\psi, \hat{A}\phi) \neq (\hat{A}\psi, \phi)$, biorthogonality *does not* hold with respect to the usual inner product, and we do not have EP self-orthogonality.

To see this explicitly, begin with two (nearby) solutions of the wave equation $\psi_{1,2}$ with eigenvalues $\omega_{1,2}$ and with incoming boundary conditions. Consider the integral

$$(\psi_2, \{\nabla^2 + \varepsilon\omega_1^2\}\psi_1) = 0. \quad (S1)$$

By integrating by parts twice, applying the boundary conditions, and dividing by a common factor of $(\omega_2 - \omega_1)$ we have

$$c_0 \hat{s}_1 \cdot \hat{s}_2 = -i(\omega_2 + \omega_1) \int dx \psi_2 \varepsilon \psi_1, \quad (\text{S2})$$

where $c_0^2 = [\psi_1^2(-L) + \psi_1^2(L)][\psi_2^2(-L) + \psi_2^2(L)]$, and $\hat{s}_{1,2} \propto (\psi_{1,2}(-L), \psi_{1,2}(L))$ are the normalized S -matrix eigenvectors at $\omega_{1,2}$ with eigenvalue equal to zero.

The dielectric function can be parametrically deformed to bring about an accidental degeneracy (EP), so that $\omega_2 \rightarrow \omega_1 \equiv \omega_0$, and $\psi_2 \rightarrow \psi_1 \equiv \psi_0$, in which case

$$c_0 \hat{s}_0 \cdot \hat{s}_0 = -2i\omega_0 \int dx \psi_0 \varepsilon \psi_0. \quad (\text{S3})$$

This is the identity used in Eq. 2 of the main text.

One may recover biorthogonality in open systems, and therefore EP self-orthogonality, by modifying the usual inner-product. For example, the new inner product defined by the subtraction of the two sides of Eq. (S2):

$$((\psi_2, \psi_1)) \equiv \int_{-L}^L dx \psi_2 \varepsilon \psi_1 - \frac{ic_0 \hat{s}_2 \cdot \hat{s}_1}{\omega_2 + \omega_1}, \quad (3)$$

makes eigenstates biorthogonal by construction, and EP self-orthogonality follows. This is the procedure followed in [1–3]. Alternatively, any field or coordinate transformation which suppresses the boundary terms of Eq. (S2) also leads to biorthogonality with respect to the ordinary inner product, provided the transformation is appropriately applied to ε , as is done in [4, 5] through perfectly matched layers or the complex scaling technique. In this way one may still speak of biorthogonal resonant or CPA states, and self-orthogonal EPs, but this is not a useful convention to adopt here: it obscures the important point that an EP of the wave operator is generally not an EP of S .

S3. COINCIDENCE OF EPS OF S AND H IN TCMT FOR SYMMETRIC OUTCOUPLING

In TCMT, the S -matrix is related to an effective Hamiltonian H (not necessarily hermitian) by

$$S = [1 - 2iW^\dagger \frac{1}{\omega - (H - iWW^\dagger)} W] S_0, \quad (\text{S4})$$

where S_0 is the “background” scattering matrix, i.e. S in the absence of resonances, and W_{ij} is a matrix of coupling coefficients between mode i and asymptotic channel j .

In the case where there are as many modes as there are asymptotic channels, W is square. Additionally, if each mode couples to exactly one distinct channel, and all outcoupling rates are identical, then $W = \sqrt{\gamma/2} \mathbb{I}$. Then equation (S4) reduces to

$$SS_0^{-1} = 1 - \frac{i\gamma}{(\omega + i\gamma/2) - H}, \quad (\text{S5})$$

Now we apply a perturbation which tunes H to a non-hermitian degeneracy. Then we can generally write $H = \Omega_{\text{EP}} + N$, where Ω_{EP} is the perturbed frequency, still degenerate, and N is nilpotent: $N \neq 0$ but $N^2 = 0$. We can expand the denominator as a geometric series, which truncates at N^2 :

$$\left[\mathbb{I} - \frac{N}{(\omega - \Omega_{\text{EP}} + i\gamma/2)} \right]^{-1} = \mathbb{I} + \frac{N}{(\omega - \Omega_{\text{EP}} + i\gamma/2)} \quad (\text{S6})$$

so that

$$SS_0^{-1} = D - \frac{i\gamma}{(\omega - \Omega_{\text{EP}} + i\gamma/2)^2} N, \quad (\text{S7})$$

where D is some diagonal matrix. This makes SS_0^{-1} manifestly exceptional.

For one-dimensional structures, such as the structures in Fig. 1 in the main text, there is no non-resonant coupling of left and right channels, and $S_0 \propto \mathbb{I}$, so that if SS_0^{-1} has an EP, then so too must S . Therefore in the geometry of Fig. 1g-i, with symmetric outcoupling, an EP of the wave operator (in the TCMT approximation this means an EP of H) implies a simultaneous EP of S .

S4. CALCULATION OF SCATTERING AMPLITUDES AT CPA EP

A. TCMT for azimuthal perturbation

In this section we derive the scattering coefficients for the waveguide-coupled microdisk at CPA EP using the coupled-mode framework.

First we consider a pair of degenerate modes of the unperturbed disk, clockwise (CW) and counterclockwise (CCW), which have angular momentum quantum numbers $-q$ and q , respectively. The degenerate complex frequency of the modes is Ω_0 . Additionally, each mode couples to one asymptotic channel of the waveguide with the same rate γ : CW to the right channel, and CCW to the left, so that in Eq. (S4), $W = \text{diag}(\sqrt{\gamma/2}, \sqrt{\gamma/2})$. The waveguide is perfectly transmitting in the absence of the pair of resonances, so the non-resonant scattering matrix is

$$S_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Upon right-multiplying both sides of Eq. (S4) by S_0 , we get a relation for the scattering amplitudes:

$$\begin{pmatrix} t & r_L \\ r_R & t \end{pmatrix} = 1 - \frac{i\gamma}{\omega - (H - i\gamma/2)} = \frac{\omega - (H + i\gamma/2)}{\omega - (H - i\gamma/2)}. \quad (\text{S8})$$

Since we have not yet specified H , this applies to both the disk with and without the grating perturbation, though with different Hamiltonians.

If we bring H to an EP by tuning parameters, we can apply Eq. (S6) so that

$$\begin{pmatrix} t & r_L \\ r_R & t \end{pmatrix} = \frac{\delta - i(\gamma - \Gamma)/2}{\delta + i(\gamma + \Gamma)/2} - \frac{i\gamma}{(\delta + i(\gamma + \Gamma)/2)^2} N \quad (\text{S9})$$

where δ is the detuning, and Γ the overall loss rate:

$$\delta = \omega - \text{Re}\{\Omega_{\text{EP}}\} \quad \Gamma/2 = -\text{Im}\{\Omega_{\text{EP}}\}.$$

Eq. (S9) fully characterizes the reflection and transmission coefficients as functions of frequency near a CPA EP, in terms of the nilpotent matrix N .

We now turn our attention to the calculation of this matrix in terms of a perturbation applied to the microdisk. Under a perturbation V the eigenvalues of a degenerate effective Hamiltonian H shift by $\delta\Omega$: $H \rightarrow H + V$, $\Omega_0 \rightarrow \Omega_0 + \delta\Omega$. On the other hand, when the wave operator $\hat{A} = -\varepsilon^{-1}\nabla^2$ is perturbed by $\delta\hat{A}$, its spectrum shifts as $\Omega_0^2 \rightarrow (\Omega_0 + \delta\Omega)^2 \simeq \Omega_0^2 + 2\Omega_0\delta\Omega$. It follows that small perturbations in the effective Hamiltonian and the wave operator are related by $V = \delta\hat{A}/2\Omega_0$.

For the case of the microdisk, we will first limit ourselves to separable perturbations $\varepsilon \rightarrow [1 + \rho(r)\tau(\theta)]\varepsilon$, for which $\delta\hat{A} = -\rho(r)\tau(\theta)\hat{A}$. The perturbation V , in the basis of the unperturbed Hamiltonian, and using the original eigenvalue equation $\hat{A}\psi = -\Omega_0^2\psi$, is

$$V_{mn} = \frac{\Omega_0}{2} \int d^2x \phi_m(\mathbf{x})\rho(r)\tau(\theta)\psi_n(\mathbf{x}). \quad (\text{S10})$$

The operator \hat{A} is symmetric, therefore the sets of left and right eigenfunctions ($\{\phi\}$, $\{\psi\}$, respectively) are equal and biorthogonal with weight ε , i.e. $\int d^2x \phi_i \varepsilon \psi_j \propto \delta_{ij}$, usually written $(\phi_j, \psi_i) = \delta_{ij}$. The eigenfunctions of \hat{A} for the unperturbed microdisk are $\psi_m(r, \theta) = R_m(r) \exp(im\theta)$, and so by biorthogonality $\phi_m = \psi_{-m}$. The matrix elements given by Eq. (S10) can be evaluated in terms of the Fourier components of $\tau(\theta)$:

$$V_{mn} = \Omega_0 C_{mn} \tau_{m-n}, \quad (\text{S11})$$

where $C_{mn} = \pi \int_0^\infty dr r R_m R_{-n} \rho$ and $\tau(\theta) = \sum_n \tau_n e^{in\theta}$. The effective Hamiltonian of the perturbed disk, in the degenerate CW/CCW basis, is therefore

$$H_{mn} = \Omega_0 \delta_{mn} + \Omega_0 C_{mn} \tau_{m-n}. \quad (\text{S12})$$

To relate this to S in Eq. (S9), we make the assignment $\Omega_{\text{EP}} = \Omega_0(1 + C\tau_0)$, and $N_{mn} = (1 - \delta_{mn})\Omega_0 C_{mn} \tau_{m-n}$. For the disk, the radial functions R_m are given by Bessel functions of integer order, so that R_m and R_{-m} are related by a phase factor, and therefore so too are $C_{m,m}$ and $C_{-m,-m}$. Therefore the C 's cannot be used to make $N^2 = 0$ with $N \neq 0$. To achieve this, it is instead necessary that exactly one of $\tau_{\pm 2q} = 0$. This requires that $\tau(\theta) \notin \mathbb{R}$, otherwise $\tau_m = \tau_{-m}^*$ and both τ 's would

vanish. This is where non-hermiticity is important for EP. Without loss of generality, take $\tau_{-2q} = 0$, so that $N_{q,-q} = \tau_{2q} \neq 0$, with all other elements of N vanishing. Plugging this into Eq. (S9) and requiring CPA ($\gamma = \Gamma$), we determine the lineshapes of the reflection and transmission coefficients at CPA EP:

$$t(\delta) = \frac{\delta}{\delta + i\Gamma}, \quad r_L(\delta) = 0$$

$$r_R(\delta) = \frac{i}{\Gamma} \frac{\Omega_0 C_{qq} \tau_{2q}}{(1 - i\delta/\Gamma)^2} = \frac{r(0)}{(1 - i\delta/\Gamma)^2} \quad (\text{S13})$$

where $\Gamma = -2\text{Im}\{\Omega_0(1 + C_{qq}\tau_0)\}$. The amplitudes for transmission and reflection in the ‘‘correct’’ direction vanish exactly as they would for CPA or critical coupling in the absence of an EP. The remaining reflection amplitude for the ‘‘wrong’’ direction of incidence, at the CPA EP frequency ($\delta = 0$), is

$$r_R(0) = -\frac{i}{2} \frac{C_{qq} \tau_{2q} (2 + iQ_0^{-1})}{2\text{Im}\{C_{qq}\tau_0\} + Q_0^{-1}(1 + \text{Re}\{C_{qq}\tau_0\})}. \quad (\text{S14})$$

Q_0 is the quality factor of the bare disk, without grating or waveguide: $Q_0 = -\text{Re}\{\Omega_0\}/2\text{Im}\{\Omega_0\}$. In the limit where the bare disk resonances have $Q_0 \gg 1$ (which is typical for WGMs), we can neglect the Q_0^{-1} terms. In this limit we also approximate the radial integral C_{qq} to be real. Hence the nontrivial reflection amplitude in the high- Q limit takes the remarkably simple form

$$|r_R(0)|^2 = \frac{1}{4} \frac{|\tau_{2q}|^2}{\text{Im}\{\tau_0\}^2}. \quad (\text{S15})$$

The overall gain/loss added to the system is encoded in τ_0 , which is therefore determined by the critical coupling condition.

The analysis can be extended to include non-separable perturbations, so long as they can be decomposed into separable pieces: $\delta\varepsilon(r, \theta) = \varepsilon \sum_j \rho^j(r) \tau^j(\theta)$. The nilpotent matrix becomes $N_{mn} = (1 - \delta_{mn})\Omega_0 \sum_j C_{mn}^j \tau_{m-n}^j$, where $C_{mn}^j = \pi \int_0^\infty dr r R_m R_{-n} \rho^j$ and $\tau^j(\theta) = \sum_n \tau_n^j e^{in\theta}$. The condition for N nilpotent is that only one of $N_{\pm q, \mp q}$ vanish, say $N_{-q,q}$: $\sum_j C_{-q,q}^j \tau_{-2q}^j = 0$, but $\sum_j C_{q,-q}^j \tau_{2q}^j \neq 0$. In this case we no longer need a non-hermitian perturbation to achieve EP, though we must rely on the radial integrals (C 's) being complex. The point scatterers used in Fig. 2 exemplify this: a purely real, (approximately) separable set of perturbations that support EP.

B. Engineering for maximal asymmetry of reflection and absorption

It is evident from Eq. (S15) that scattering from the waveguide-disk system is entirely characterized by the

two Fourier components of the perturbation τ_0 and τ_{2q} , which suggests that the appropriate design to consider is a non-hermitian grating. We consider only gratings with no gain, with alternating regions of loss and no loss. The simplest experimentally feasible azimuthal grating of this type is piecewise constant, whose real and imaginary parts have the same angular width ϕ and periodicity $2\pi/P$ ($P \in \mathbb{Z}$), and an angular offset χ between them:

$$\tau(\theta) = f(\theta) + if(\theta - \chi), \quad (\text{S16})$$

where $f(0 < \theta < \phi) = c$, $f(\phi < \theta < 2\pi/P) = 0$, and $f(\theta + 2\pi/P) = f(\theta)$, for some constant c . The angular offset χ is determined from $\tau_{-2q} = (1 + ie^{2iq\chi})f_{2q} = 0$, which implies

$$\chi = (M - 1/4)\pi/q. \quad (\text{S17})$$

Of course had we demanded the $+2q$ component to vanish, this would be $\chi = (M + 1/4)\pi/q$.

We can express the reflection from the ‘‘incorrect’’ side in terms of f_m according to Eq. (S15), and using Eq. (S17):

$$r_{\text{R}}(0) = \left| \frac{f_{2q}}{f_0} \right|^2 \quad (\text{S18})$$

The Fourier components of f vanish for m not equal to a multiple of P ; the non-vanishing components satisfy

$$f_{n \cdot P} = c \frac{P}{2\pi} \int_0^\phi d\theta e^{-inP\theta} = e^{-inP\phi/2} \frac{c}{n\pi} \sin \frac{nP\phi}{2}.$$

Plugging this into Eq. (S18) gives

$$r_{\text{R}}(0) = \left| \frac{\sin q\phi}{q\phi} \right|^2, \quad (\text{S19})$$

so long as $NP = 2q$, where N is the order of the grating that we are using to couple the $\pm q$ modes.

We see that the asymmetry of the reflection, and therefore of the absorption, achieves its maximal value, unity, for thin gratings ($\phi \rightarrow 0$). The intuition is that the lossy regions can be ‘‘hidden’’ in the nodes of the back-scattered field when excited from the non-CPA side, and the thinner they are, the better they are hidden. Since the field is a running wave when excited from the CPA side, the material loss is just as effective regardless of how narrow its spatial distribution. This is evident in Fig. 3.

A more general type of grating has different widths, contrasts, and periodicities for its real and imaginary parts. If the real part has contrast a , periodicity L , width ϕ , while the imaginary part has (b, P, ψ) , the conditions for CPA EP are

$$\left| \frac{\sin q\phi}{\sin q\psi} \right| = \frac{bP}{aL},$$

L and P must both divide $2q$, and the offset is

$$\chi = (M - 1/4)\pi/q + (\phi - \psi)/2.$$

In this case, the asymmetric reflection is

$$r_{\text{R}}(0) = \left| \frac{\sin q\phi}{q\phi} \right|^2 \cos^2(q[\phi - \psi]),$$

which shows that the more restrictive grating analyzed earlier ($\phi = \psi$) is optimal.

It is worth noting that gauged \mathcal{PT} -symmetry corresponds to $\phi = \pi/P$, which yields $|r_{\text{R}}(0)|^2 = \text{sinc}^2(N\pi/2) < 41\%$.

S5. FREE SPACE LOSS AND CHIRAL CPA EP

The disk plus waveguide does not admit CPA solutions which only propagate in on the waveguide; the exact CPA solutions will require some small flux to excite the disk from free space, just as the corresponding laser would radiate weakly into those free space channels. If we simply take the system at the CPA EP solution parameters but excite solely through the waveguide, we then do not expect to find 100% absorption or zero transmission along the fiber; and indeed when we implemented this procedure we found a small, but measurable transmission. Since we are interested in a chiral absorber without free-space excitation we hence adjusted the waveguide parameters in order to minimize this transmission, moving away from the exact CPA EP point.

Qualitatively we expect free-space channels to act as a small additional loss with respect to the guided channels. Therefore we increased the coupling to the fiber by a few percent until the transmission in the fiber was minimized, while otherwise maintaining the same structure, and found that the transmission became negligible. Since we are no longer solving the exact CPA problem we are no longer guaranteed that all the flux will be absorbed in the disk with grating, some will be lost to free space radiation. This is the reason why in Fig. 3 the absorption from the CPA EP is not unity, but it is still greater than 97%.

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