Theory of the Spatial Structure of Non-linear Modes in Novel and Complex Laser Cavities

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ABSTRACT
A new formalism [1,2] for calculating exact steady-state non-linear multi-mode lasing states for complex resonators is developed and applied to conventional edge-emitting lasers and to lasers with chaotic or random cavities. The theory solves a long-standing problem in lasing theory: how to describe the multi-mode lasing states of an open cavity. Moreover it includes the effects of mode competition and spatial hole-burning to all orders within the approximation of stationary inversion. Lasing modes are expanded in terms of sets of biorthogonal "constant flux" (CF) states and satisfy a self-consistent equation. For high finesse cavities each lasing mode is proportional to one CF state which inside the cavity behaves like a linear resonance; for low finesse as in a random laser, novel composite modes are predicted which do not correspond to any passive cavity resonance.
Keywords: lasers, spatial hole-burning, multimode, random lasers, Maxwell-Bloch equations.

1. INTRODUCTION

The nature of the electric field in a laser well above threshold has been a long-standing question in laser theory, complicated by the difficulty of treating exactly both the non-linear interaction and the openness of a laser cavity. Recently the authors have proposed a framework for solving this problem based on the solution of a set of non-linear integral equations for the lasing modes [1,2] and have applied it to a simple low-finesse edge-emitting laser [2]. In the current short paper we describe the basic concepts of the new approach, their application to conventional lasers, and the initial stages of an application to random lasers. To our knowledge this will be the first calculation of the lasing modes of a realistic coherent feedback random laser [3].

2. SELF-CONSISTENT EQUATION FOR MULTI-MODE LASING

In refs [1,2] we considered the semiclassical Maxwell-Bloch equations in the standard rotating wave and slowly varying envelope approximations for a medium with uniform gain confined in a background (host medium) of arbitrary spatially varying index of refraction, $n^2(x) = \varepsilon(x)$. We assumed a general multi-periodic form for the electric field $e(x,t) = \sum_\mu \Psi_\mu(x) e^{-i\Omega_\mu t}$, made the rather general approximation of stationary inversion, and showed that the non-linear lasing modes we were looking for satisfy a set of self-consistent integral equations,

$$\Psi_\mu(x) = i\frac{D_0\gamma_\perp}{\gamma_\perp - ik_\mu}\int_{cavity} dx' G(x,x';\Omega_\mu)\frac{\Psi_\mu(x')}{\varepsilon(x')\left(1+\sum_\nu g_\nu \mid \Psi_\nu(x')\mid^2\right)} \tag{1}$$

where $g_\nu = g(\Omega_\nu)$ is the gain profile evaluated at the lasing frequency, D_0 is the scaled pump strength, $\gamma_\perp$ is the transverse relaxation rate of the atomic line and g is the dipole matrix element (henceforth we will use only wavevector k_μ and not frequency). G is the Green function of the cavity wave equation

$$\left[(k_a + k)^2 + \frac{1}{\varepsilon(x)}\partial_x^2\right]G(x,x') = \delta(x - x') \tag{2}$$

($k_a = \omega_a/c$ where ω_a is the atomic transition frequency),with spectral representation

$$G(x,x') = \sum_m \frac{\varphi_m(x)\overline{\varphi}_m^*(x')}{\eta_m\left[(k_a+k)^2 - (k_a+k_m)^2\right]} \tag{3}$$

This Green function satisfies the non-hermitian boundary condition that it contains only outgoing waves at infinity [1]. Due to this the functions in the numerator of its spectral representation form a dual set of biorthogonal linear solutions of the homogeneous cavity wave equation [1] with only outgoing waves (φ_m) or incoming $(\overline{\varphi}_m)$ waves at infinity. These solutions, which we refer to as constant flux (CF) states, have complex-wavevectors (amplifying or absorbing) inside the cavity and the real (flux-conserving) wavevector k_μ outside the cavity. The set $\{\varphi_m\}$ is complete and can be used to express any lasing mode, $\Psi_\mu(x) = \sum a_m^\mu \varphi_m^\mu(x)$. Note that these states are not the resonances of the cavity, which have complex wavevectors even outside the cavity and are not an appropriate set for expressing the lasing state. In ref. [1] it was shown that when the cavity has high finesse a single CF state represents the lasing mode, but in a low finesse cavity [2] we find that several CF states contribute to the lasing state, particularly far above threshold. This new method for calculating exact non-linear lasing states should be useful across the range of complex cavity lasers currently being studied: wave-chaotic microcavity lasers [4-6], random lasers [3,7,8] and photonic bandgap lasers [9,10]. By expanding in CF states and truncating over some reasonable range of wavevectors the self-consistent lasing equation can be expressed in terms of the complex vector of components of the CF states, a^μ, in the form [2]:

$$a^\mu = D_0 F(\{a^\nu\},\{k_\mu\}) \tag{4}$$

where F is an infinite-order non-linear function of $\{a^\nu\}$ This defines a non-linear map of the vector a^μ whose fixed points are the lasing solutions. Note that the map of a^μ depends on all the other non-zero a^ν, reflecting the effects of modal interactions and spatial hole-burning.

3. RESULTS FOR 1D EDGE-EMITTING LASER

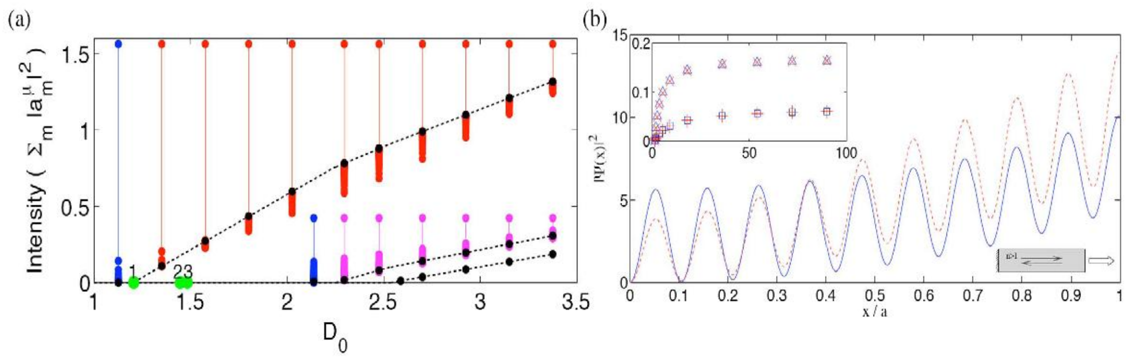

Figure 1. (a) Convergence and solution of the multimode lasing map for 1D edge emitting laser resonator of length $a = 1$, index $n_0 = 1.5$, atomic frequency $k_a = 19.89$ and gain width $\gamma_\perp = 4.0$ vs. pump D_0. Three modes lase in this range. At threshold they correspond to CF states $m = 8,9,10$ with threshold lasing frequencies $k_t^{(8)} = 18.08, k_t^{(9)} = 19.91$, and $k_t^{(10)} = 21.76$, and non-interacting thresholds $D_0^{(9)} = 1.204$, $D_0^{(10)} = 1.445$, $D_0^{(8)} = 1.482$ (green dots). $k_t^{(9)} \approx k_a$ and $m = 9$ and thus has the lowest threshold. Due to mode competition, modes 2,3 do not lase until much higher values ($D_0 = 2.25, 2.53$). Each mode is represented by an 11 component vector of CF states; we plot the sum of $|a_m|^2$ vs. pump D_0. Below threshold the vectors flow to zero (blue dots). For $D_0 \geq 1.204$ the sum flows (red dots) to a non-zero value (black dashed line), and above $D_0 = 2.25, 2.53$, two additional non-zero vector fixed points (modes) are found (convergence only shown for modes 1,2). (b) Non-linear electric field intensity for this laser in the single-mode regime ($\gamma_\perp = 0.5, D_0 = 9$.) The full field (red line) has an appreciably larger amplitude at the output $x = a$ than the "single-pole" approximation (blue) which neglects the sideband CF components. Upper inset: The ratio of the two largest CF sideband components to that of the central pole for $n_0 = 1.5$ (□, ×) and $n_0 = 3$ (□, +) vs. pump strength D_0. Lower inset: schematic of the edge-emitting laser cavity.

To test this new formalism we apply it first to a simple one-dimensional edge-emitting laser consisting of a perfect mirror at the origin and a dielectric interface as *x*=*a*. Fig. 1 a shows the calculation of the vectors a by iterative solution of this equation for this 1D uniform index laser [2]. We are able to find the exact linear (saturation) behavior of the output intensity far above threshold. Our results show that mode competition has a large effect on the higher thresholds, as the thresholds for the 2nd and 3rd mode are increased by factors of 4 or more. The solution vector is dominated by three CF states, one "central pole" and the two nearest in frequency on each side (spatial "sidebands"); 1 b shows that neglecting these sidebands leads to a 50% underestimate of the output power of the mode well above threshold.

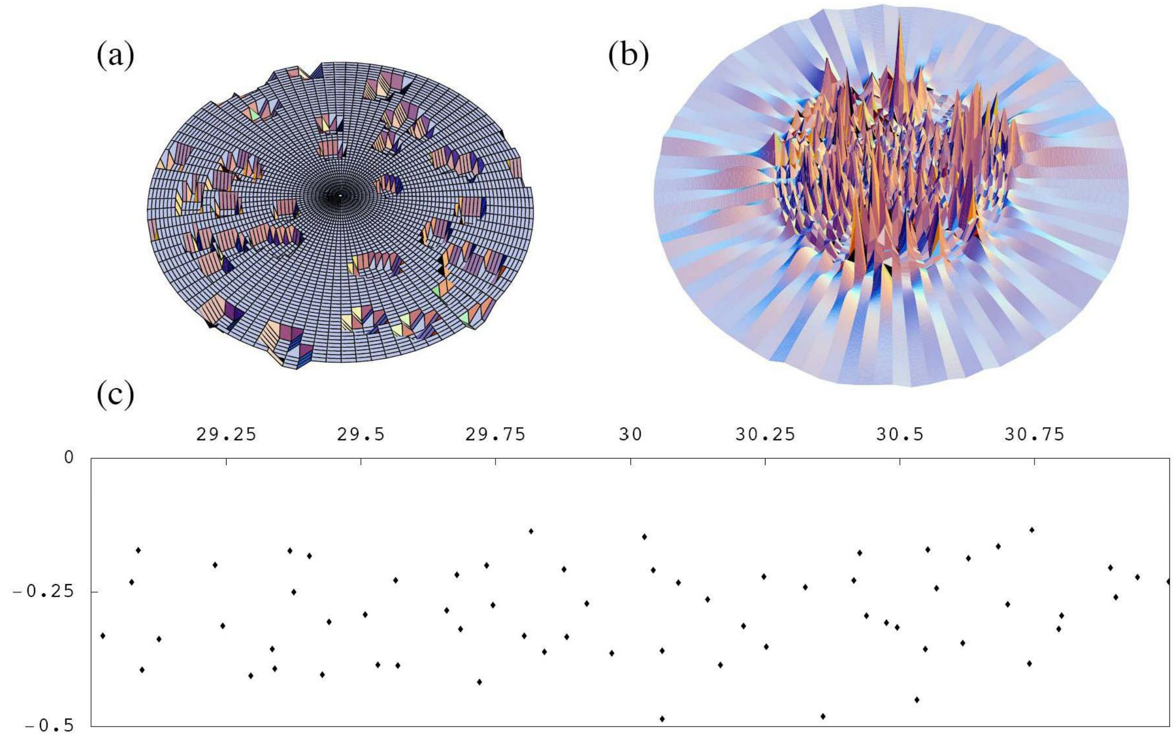

Figure 2. Microdisk laser with index of refraction $n_0 = 1.5$ in which voids with $n = 1$ are placed randomly. The radius of the disk is $R = 1$ and the external frequency $k = 30$. (a) Configuration of refractive index n in the disk. The black grid indicates the employed polar-coordinate discretization.[7] (b) The intensity of a constant-flux state corresponding to the eigenvalue $k_m = 30.042 - 0.209i$ is shown in the radial interval $r \in [0, 2R]$. (c) Distribution of CF-eigenvalues k_m in an interval around $\mathrm{Re}(k_\mu)$. Note the mean eigenvalue spacing in the interval is ≈ 0.03 while the mean value of the absolute value of the imaginary part $\mathrm{Im}(k_m) < 0$ is ≈ 0.3 so the random laser cavity has fractional finesse $f \approx 0.1$.

4. RANDOM LASERS

A major puzzle has been how to understand the lasing modes of a random cavity in the diffusive (delocalized) regime when the linear problem has no distinct resonances (cavity finesse f is parametrically smaller than unity), yet the lasing mode has a narrow line and Poisson photon statistics well above threshold [3,7]. Our formalism suggests that in this case $1/f > 1$ CF states contribute almost equally to the lasing state, which thus does not correspond to any single solution of the linear wave equation, but is instead a "composite" mode. To demonstrate this we have defined the following model: a circular disk of gain material of radius R contains a randomly distributed group dielectric "particles" of a certain density. (This is similar to the ZnO nanoparticle clusters studied by Cao, *et al.* [3] except that we take the clusters to terminate on a circular boundary and assume gain everywhere in the cluster for convenience). The non-hermitian CF boundary condition is then expressed by the condition that at the cluster boundary the solution is continuous and may be expressed as a superposition of only *outgoing* Hankel functions of wavevector k_μ. This calculation can be discretized and expressed as a non-symmetric eigenvalue problem [11]. An example of a single random CF state and a typical CF spectrum are shown in Fig. 2. It is evident that we are in the regime of fractional finesse (resonance width larger than spacing). Thus we expect lasing modes to be superpositions of many such random CF states, the "composite modes" discussed above.

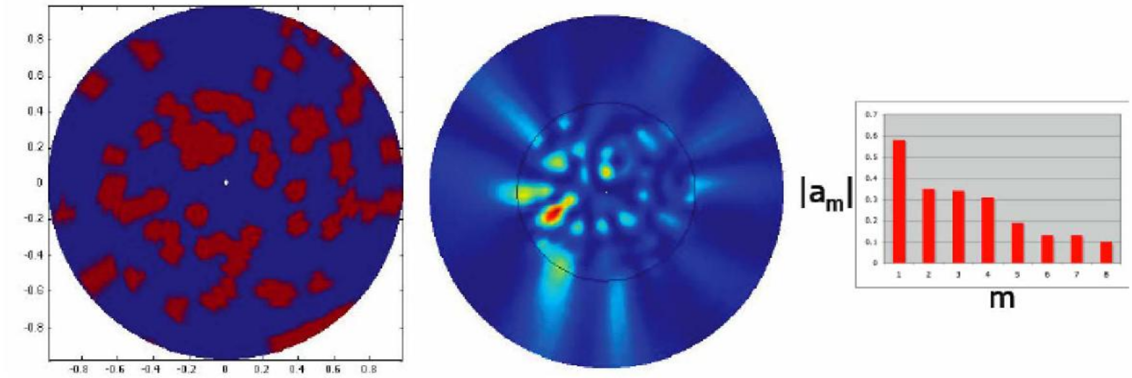

Figure 3. Left: distribution of index $n = 1.5$ *"particles" defining the random laser cavity. Center: false color representation of the random lasing state. Right: Distribution of components* $|a_m|$ *of the 8 CF states which contribute substantially to this lasing solution.* $= 0.5, k_a = 10, k_\mu = 10.0725$ *in units of the inverse cavity radius* $(r = 1)$, $D_0 = 40.0$.

Recently we have succeeded in implementation and solution of the self-consistent equation (1) for this model. An example of single-mode random lasing is given in Fig. 3 above. Eight CF states contribute significantly to the random lasing mode at a pump strength three times the threshold pump strength, illustration the emergence of composite modes in a laser cavity with no sharp resonances at all (fractional finesse). Future work will analyze the statistical properties of the first and higher thresholds and of the lasing states in space and in frequency.

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