

# Theory of the Spatial Structure of Non-linear Modes in Conventional and Random Lasers

Hakan E. Türeci,<sup>1</sup> Li Ge,<sup>2</sup> Stefan Rotter,<sup>2</sup> and A. Douglas Stone<sup>2</sup>

<sup>1</sup>*Institute of Quantum Electronics, ETH Zürich, CH-8093 Zürich, Switzerland*

<sup>2</sup>*Department of Applied Physics, P.O. Box 208284, Yale University, New Haven, CT 06520-8284, USA*  
 tureci@phys.ethz.ch

**Abstract:** A new formalism for calculating exact non-linear multi-mode lasing states for complex resonators is applied to a conventional edge-emitting laser and a 2D random laser. Novel "composite" random lasing states are expected. © 2007 Optical Society of America

**OCIS codes:** 350.3950, 270.3430, 140.3410

The nature of the electric field in a laser well above threshold has been a long-standing question in laser theory, complicated by the difficulty of treating exactly both the non-linear interaction and the openness of a laser cavity. Recently the authors have proposed a framework for solving this problem exactly based on the solution of a set of non-linear integral equations for the lasing modes<sup>1,2</sup> and have applied it to a simple low-finesse edge-emitting laser.<sup>2</sup> In the current short paper we describe the basic concepts of the new approach, their application to conventional lasers, and the initial stages of an application to random lasers. To our knowledge this will be the first calculation of the lasing modes of a realistic coherent feedback random laser.<sup>3</sup>

In refs. 1,2 we considered the semiclassical Maxwell-Bloch equations in the standard rotating wave and slowly varying envelope approximations for a medium with uniform gain confined in a background (host

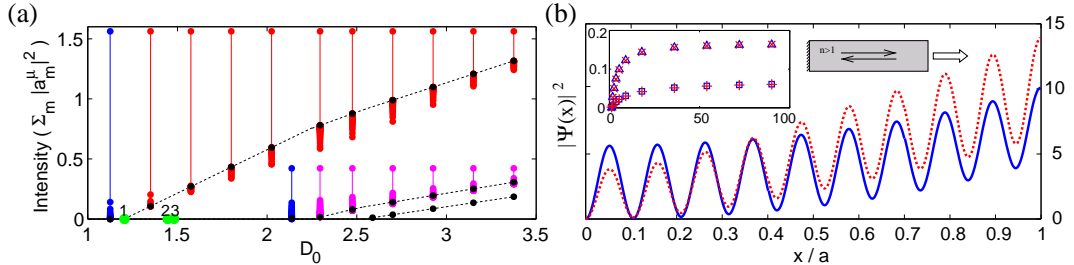


Fig. 1. (a) Convergence and solution of the multimode lasing map for 1D edge emitting laser resonator of length  $a = 1$ , index  $n_0 = 1.5$ , atomic frequency  $k_a = 19.89$  and gain width  $\gamma_{\perp} = 4.0$  vs. pump  $D_0$ . Three modes lase in this range. At threshold they correspond to CF states  $m = 8, 9, 10$  with threshold lasing frequencies  $k_t^{(8)} = 18.08$ ,  $k_t^{(9)} = 19.91$ , and  $k_t^{(10)} = 21.76$ , and non-interacting thresholds  $D_0^{(9)} = 1.204$ ,  $D_0^{(10)} = 1.445$ ,  $D_0^{(8)} = 1.482$  (green dots). Since  $k_t^{(9)}$  is closest to  $k_a$ , mode  $m = 9$  thus has the lowest threshold. Due to mode competition, modes 2,3 do not lase until much higher values ( $D_0 = 2.25, 2.53$ ). Each mode is represented by an 11 component vector of CF states; we plot the sum of  $|a_m|^2$  vs. pump  $D_0$ . Below threshold the vectors flow to zero (blue dots). For  $D_0 \geq 1.204$  the sum flows (red dots) to a non-zero value (black dashed line), and above  $D_0 = 2.25, 2.53$ , two additional non-zero vector fixed points (modes) are found (convergence only shown for modes 1,2). (b) Non-linear electric field intensity for this laser in the single-mode regime ( $\gamma_{\perp} = 0.5, D_0 = 9$ ). The full field (red line) has an appreciably larger amplitude at the output  $x = a$  than the "single-pole" approximation (blue) which neglects the sideband CF components. Left inset: The ratio of the two largest CF sideband components to that of the central pole for  $n_0 = 1.5$  ( $\square, \times$ ) and  $n_0 = 3$  ( $\square, +$ ) vs. pump strength  $D_0$ . Right inset: schematic of the edge-emitting laser cavity.

medium) of arbitrary spatially varying index of refraction,  $n(\mathbf{x})$ . We assumed a general multi-periodic form for the electric field  $e(\mathbf{x}, t) = \sum_{\mu} \Psi_{\mu}(\mathbf{x})e^{-i\Omega_{\mu}t}$  and showed that the stationary non-linear lasing modes we were looking for satisfy a set of self-consistent integral equations,

$$\Psi_{\mu}(\mathbf{x}) = \frac{i\frac{4\pi k_a^2 d^2}{\hbar} \tilde{D}_0}{-i\Omega_{\mu} + \gamma_{\perp}} \int_{\mathcal{D}} d\mathbf{x}' \frac{G(\mathbf{x}, \mathbf{x}'|\Omega_{\mu})\Psi_{\mu}(\mathbf{x}')}{1 + \sum_{\nu} g(\Omega_{\nu})|\Psi_{\nu}(\mathbf{x}')|^2}, \quad (1)$$

where  $g(\Omega_{\mu})$  is the gain profile evaluated at the lasing frequency,  $\mathcal{D}$  is the cavity domain and  $G$  is the Green function of the cavity wave equation  $[\nabla^2 + n^2(\mathbf{x})k_{\mu}^2]G(\mathbf{x}, \mathbf{x}'|\omega) = \delta^3(\mathbf{x} - \mathbf{x}')$  with purely outgoing boundary conditions and  $k_{\mu} = \Omega_{\mu}/c$  is the wavevector of the lasing solution at infinity. The remaining variables are:  $k_a = \omega_a/c$  where  $\omega_a$  is the atomic transition frequency,  $\gamma_{\perp}$  is the homogeneous broadening of the gain medium,  $d$  is the dipole matrix element, and  $\tilde{D}_0$  is the pump strength.

This non-hermitian outgoing cavity Green function can be expressed in a spectral representation in terms of a dual set of biorthogonal linear modes,<sup>1</sup> which correspond to complex-wavevector solutions inside the cavity and flux-conserving outgoing waves of wavevector  $k_{\mu}$  outside the cavity. This set is complete and can be used to express any lasing mode,  $\Psi_{\mu}(\mathbf{x}) = \sum a_m^{\mu} \varphi_m^{\mu}(\mathbf{x})$ . We refer to these biorthogonal states as constant-flux (CF) states. Note that these states are not the resonances of the cavity, which have complex wavevectors outside the cavity and are not an appropriate set for expressing the lasing state. In ref. 1 it was shown that when the cavity has high finesse a single CF state represents the lasing mode, but in a low finesse cavity<sup>2</sup> we find that several CF states contribute to the lasing state, particularly far above threshold. This new method for calculating exact non-linear lasing states should be useful across the range of complex cavity lasers currently being studied: wave-chaotic microcavity lasers,<sup>4,5,6</sup> random lasers<sup>3,7,8</sup> and photonic bandgap lasers.<sup>9,10</sup>

The self-consistent lasing equation expressed in terms of this vector of components of the CF states,  $\mathbf{a}^{\mu}$ , takes the rescaled form:<sup>2</sup>

$$a_m^{\mu} = \frac{iD_0\gamma_{\perp}}{(\gamma_{\perp} - ik_{\mu})} \frac{1}{(k_{\mu} - k_m^{\mu})} \int_{cavity} d\mathbf{x} \frac{\bar{\varphi}_m^{\mu*}(\mathbf{x}) \sum_p a_p^{\mu} \varphi_p^{\mu}(\mathbf{x})}{1 + \sum_{\nu,q,r} g(k_{\nu}) a_q^{\nu} a_r^{\nu*} \varphi_q^{\nu}(\mathbf{x}) \varphi_r^{\nu*}(\mathbf{x})}, \quad (2)$$

where  $\{k_m^{\mu}\}$  are the complex CF wavevectors (different for each lasing mode  $\mu$ ),  $\bar{\varphi}_m^{\mu}$  are the dual functions to the  $\varphi_m^{\mu}$  CF states and  $D_0$  is the scaled pump strength. This defines a non-linear map of the vector  $\mathbf{a}^{\mu}$  whose fixed points are the lasing solutions. Note that the map of  $\mathbf{a}^{\mu}$  depends on all the other non-zero  $\mathbf{a}^{\nu}$ , reflecting effects of modal interactions and “spatial hole-burning” and that the non-linearity is treated exactly.

Fig. 1a shows the calculation of the vectors  $\mathbf{a}$  by iterative solution of this equation for a 1D uniform index laser.<sup>2</sup> We are able to find the exact linear behavior of the output intensity far above threshold and include the effects of mode competition as the thresholds for the 2nd and 3rd mode are increased by factors of 4 or more. The solution vector is dominated by three CF states, one “central pole” and the two nearest in frequency on each side (spatial “sidebands”); Fig. 1b shows that neglecting these sidebands leads to a 50% underestimate of the output power of the mode well above threshold.

A major puzzle has been how to understand the lasing modes of a random cavity in the diffusive (delocalized) regime when the linear problem has no distinct resonances (cavity finesse  $f$  is parametrically smaller than unity), yet the lasing mode has a narrow line and Poisson photon statistics well above threshold.<sup>3,7</sup> Our formalism suggests that in this case  $g = 1/f$  CF states contribute almost equally to the lasing state, which thus does not correspond to any single solution of the linear wave equation, but is instead a “composite” mode. To demonstrate this we have defined the following model: a circular disk of gain material of radius  $R$  contains a dielectric material with randomly placed voids of a certain density (this is similar to the ZnO nanoparticle clusters of Ref.<sup>3</sup> except that we take the clusters to terminate on a circular boundary and assume gain everywhere in the cluster for convenience). The non-hermitian CF boundary condition is then expressed by the condition that at the cluster boundary the solution is continuous and may be expressed as a superposition of only *outgoing* Hankel functions of wavevector  $k_{\mu}$ . This calculation can be discretized and expressed as a generalized eigenvalue problem.<sup>11</sup> An example of a single random CF state and a typical CF spectrum are shown in Fig. 2. It is evident that we are in the regime of fractional finesse (resonance width larger than spacing). Thus we expect lasing modes to be superpositions of many such random CF states, the “composite modes” discussed above. Currently we are solving Eq. (2) for such random CF states to test this expectation and study the statistical properties of such lasing modes, which are expected to be substantially different from those of single linear diffusive modes.

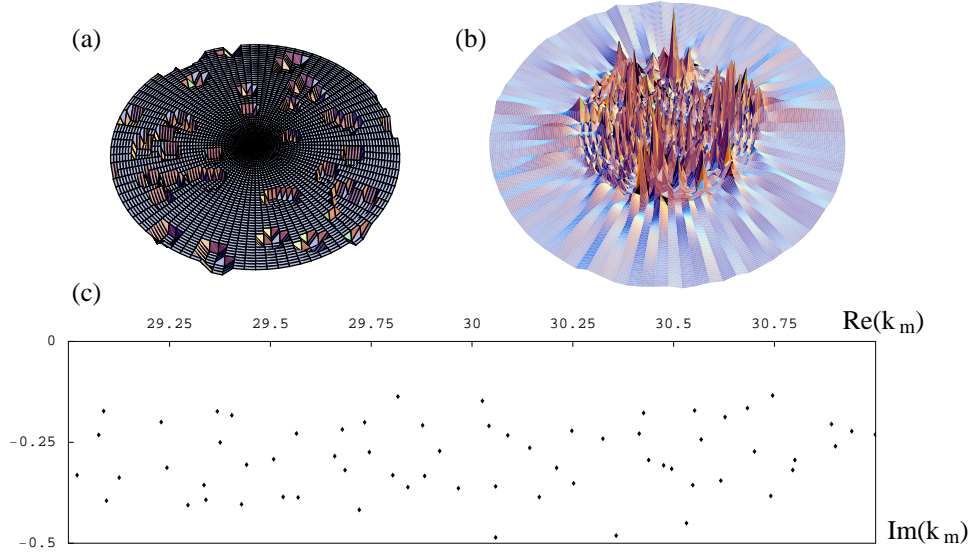


Fig. 2. Microdisk laser with index of refraction  $n_0 = 1.5$  in which voids with  $n = 1$  are placed randomly. The radius of the disk is  $R = 1$  and the external frequency  $k_\mu = 30$ . (a) Configuration of refractive index  $n(\mathbf{x})$  in the disk. The black grid indicates the employed polar-coordinate discretization. (b) The intensity of a constant-flux state corresponding to the eigenvalue  $k_m = 30.042 - 0.209i$  is shown in the radial interval  $r \in [0, 2R]$ . (c) Distribution of CF-eigenvalues  $k_m$  in an interval around  $\text{Re}(k_\mu) = 30$ . Note the mean eigenvalue spacing in the interval is  $\approx 0.03$  while the mean absolute value of the imaginary part of  $k_m$  is  $\approx 0.3$  so the random laser cavity has fractional finesse  $f \approx 0.1$ .

## 1. References

- [1] H. E. Türeci, A. D. Stone, and B. Collier, “Self-consistent multimode lasing theory for complex or random lasing media,” *Phys. Rev. A* **74**, 043822 and cond-mat/0605673.
- [2] H. E. Türeci, A. D. Stone and L. Ge, “Theory of the spatial structure of non-linear lasing modes,” submitted to PRL, cond-mat/0610229.
- [3] H. Cao, Y. Zhao, S. T. Ho, E. W. Seelig, Q. H. Wang, R. P. H. Chang, “Random Laser Action in Semiconductor Powder,” *Phys. Rev. Lett.* **82**, 2278 (1999).
- [4] C. Gmachl, F. Capasso, E.E. Narimanov, J.U. Nöckel, A. D. Stone, J. Faist, D. Sivco and A. Cho, “High power directional emission from lasers with chaotic resonators,” *Science* **280**, 1556 (1998).
- [5] H. E. Türeci, H. G. L. Schwefel, Ph. Jacquod, A. D. Stone, “Modes of wave-chaotic dielectric resonators,” *Progress in Optics* **47**, 75, Elsevier (2005).
- [6] H. G. L. Schwefel, H. E. Türeci, A. D. Stone, R. K. Chang, “Progress in Asymmetric Resonant Cavities: Using Shape as a Design Parameter in Dielectric Microcavity Lasers,” Chapter 9 in *Progress in Optical Cavities*, K.J. Vahala Editor, (World Scientific, 2004).
- [7] H. Cao, “Lasing in Random Media”, *Waves in Random Media*, **13**, R1-R39, (2003).
- [8] V. Milner and A. Z. Genack, “Photon localization laser: Low-threshold lasing in a random amplifying layered medium via wave localization,” *Phys. Rev. Lett.* **94**, 073901 (2005).
- [9] O. Painter, R. K. Lee, A. Scherer, A. Yariv, J. D. O’Brien, P. D. Dapkus, and I. Kim, “Two-dimensional Photonic Band-Gap Defect Mode Laser,” *Science* **284**, 1819 (1999).
- [10] H. Y. Ryu, S. H. Kim, H. G. Park, J. K. Hwang, Y. H. Lee, and J. S. Kim, “Square-lattice photonic band-gap single-cell laser operating in the lowest-order whispering gallery mode,” *Appl. Phys. Lett.* **80**, 3883 (2002).
- [11] H. E. Türeci, S. Rotter, L. Ge and A. D. Stone, unpublished.