Supplemental Material for "Detecting and Focusing on a Nonlinear Target in a Complex Medium"

I. THEORY

We consider a system without free charges and no magnetisation so that the wave equation is given by

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\mu_0 \partial_t^2 \boldsymbol{D}. \tag{S1}$$

We can now split up the electric displacement field $\boldsymbol{D} = \epsilon_0 \epsilon \boldsymbol{E} + \boldsymbol{P}^{\mathrm{NL}}$ into a linear part given by the relative electric permittivity ϵ and a polarization part $\boldsymbol{P}^{\mathrm{NL}}[\boldsymbol{E}]$, which is nonlinear in $\boldsymbol{E}$. We are now going to make use of the Green's tensor G_ω of the linear system at frequency ω with outgoing boundary conditions, i.e.

$$\nabla \times (\nabla \times G_\omega)(\boldsymbol{r}, \boldsymbol{r}\,') - \frac{\omega^2}{c^2} \epsilon(\boldsymbol{r}) G_\omega(\boldsymbol{r}, \boldsymbol{r}\,') = 1\delta(\boldsymbol{r} - \boldsymbol{r}\,'). \tag{S2}$$

This definition allows us to rewrite the monochromatic solution of the wave equation at frequency ω using the Green's tensor in the far field

$$\boldsymbol{E}_\omega(\boldsymbol{r}) = \boldsymbol{E}_\omega^{\mathrm{L}}(\boldsymbol{r}) + \frac{k^2}{\epsilon_0} \int G_\omega(\boldsymbol{r}, \boldsymbol{r}\,') \boldsymbol{P}_\omega^{\mathrm{NL}}[\boldsymbol{E}(\boldsymbol{r}\,')] d\boldsymbol{r}\,', \tag{S3}$$

where $k = \omega/c$, $\boldsymbol{E}_\omega^{\mathrm{L}}$ describes the incident field component at frequency ω scattered in the linear system (i.e. $\nabla \times (\nabla \times \boldsymbol{E}_\omega^{\mathrm{L}}) + k^2 \epsilon \boldsymbol{E}_\omega^{\mathrm{L}} = 0$) and $\boldsymbol{P}_\omega^{\mathrm{NL}}(\boldsymbol{E})$ the nonlinear polarization field at frequency ω corresponding to the full electric field $\boldsymbol{E}$. We want to emphasize here that $\boldsymbol{P}_\omega^{\mathrm{NL}}(\boldsymbol{E})$ depends in general on the full electromagnetic field $\boldsymbol{E}$ at all frequencies and not just the field component $\boldsymbol{E}_\omega$ at frequency ω. This occurs in cases such as second harmonic generation, where the polarization field $P_{2\omega}$ at 2ω depends on the electric field component $\boldsymbol{E}_\omega$ at ω.

A. Separating the linear and nonlinear contributions

Our first goal is going to be to separate these two contributions. For this we can make us of the fact that the linear scaling of the output field with respect to the input field is broken due to the presence of the nonlinearity. We proceed by varying the strength (quantified by α) of the incident field $\boldsymbol{E}_\omega^{\mathrm{L,in}} \to \boldsymbol{E}_\omega^{\mathrm{L,in}} \alpha$, which produces a predictable linear response in the linear contributions of the scattered field $\boldsymbol{E}_\omega^{\mathrm{L},\alpha} = \boldsymbol{E}_\omega^{\mathrm{L}} \alpha$. Therefore by varying α we can extract the nonlinear contributions of the field by removing exactly those contributions with linear scaling

$$\boldsymbol{E}_\omega^{\alpha+\Delta\alpha} - \frac{\alpha + \Delta\alpha}{\alpha} \boldsymbol{E}_\omega^{\alpha} = \frac{k^2}{\epsilon_0} \int G_\omega(\boldsymbol{r}, \boldsymbol{r}') \left\{ \boldsymbol{P}_\omega^{\mathrm{NL}}[\boldsymbol{E}^{\alpha+\Delta\alpha}(\boldsymbol{r}\,')] - \frac{\alpha + \Delta\alpha}{\alpha} \boldsymbol{P}_\omega^{\mathrm{NL}}[\boldsymbol{E}^{\alpha}(\boldsymbol{r}\,')] \right\} d\boldsymbol{r}\,' \tag{S4}$$

whereby the superscript $\alpha, \alpha + \Delta\alpha$ quantifies the strength of the incident field. The left hand side of the equation now effectively corresponds to a field that originates at the nonlinearity and propagates out of the system according to the Green's tensor of the corresponding linear system $G_\omega(\boldsymbol{r}, \boldsymbol{r}')$. This already shows that a nonlinearity can be detected just by observing the divergence of the outgoing field from the linear field strength scaling property. In order to rewrite this expression to Eq. (2) of the main text we divide by $\alpha + \Delta\alpha$ and define the difference of the normalized fields $\delta\boldsymbol{E}_\omega(\boldsymbol{r}) = (\alpha + \Delta\alpha)^{-1} \boldsymbol{E}_\omega^{\alpha+\Delta\alpha}(\boldsymbol{r}) - \alpha^{-1} \boldsymbol{E}_\omega^{\alpha}(\boldsymbol{r})$, which satisfies

$$\delta\boldsymbol{E}_\omega(\boldsymbol{r}) = \frac{k^2}{\epsilon_0} \int d\boldsymbol{r}\,' G_\omega(\boldsymbol{r}, \boldsymbol{r}\,') \delta\boldsymbol{P}_\omega^{\mathrm{NL}}(\boldsymbol{r}\,')\,, \tag{S5}$$

with $\delta\boldsymbol{P}_\omega^{\mathrm{NL}}(\boldsymbol{r}\,') = (\alpha + \Delta\alpha)^{-1} \boldsymbol{P}_\omega^{\mathrm{NL}}[\boldsymbol{E}^{\alpha+\Delta\alpha}(\boldsymbol{r}\,')] - \alpha^{-1} \boldsymbol{P}_\omega^{\mathrm{NL}}[\boldsymbol{E}^{\alpha}(\boldsymbol{r}\,')]$. In the next section we are going to see how this can be exploited to create a focusing field on a point-like target.

B. Focusing on the nonlinearity

Our next objective will be to find an incident wavefront that focuses onto the location of the nonlinearity. We reverse the outgoing finite difference field (assumed non-zero from here on) derived in the previous section so that the incident field is given by $\boldsymbol{E}_\omega^{\mathrm{opt}} = \delta \boldsymbol{E}_\omega^*$ in the far field and creates a focus at the nonlinear dielectric. This can be seen for systems, where ϵ is real valued (i.e. no absorption or gain present) by using Green's identity to identify the linear response $\boldsymbol{E}_\omega^{\mathrm{L,opt}}$ of the system to the incident field $\delta \boldsymbol{E}_\omega^*$, i.e. a system with the nonlinearity removed,

$$\boldsymbol{E}_\omega^{\mathrm{L,opt}}(\boldsymbol{r}_1) = \int_{S_R^3} [\boldsymbol{n} \times G_\omega(\boldsymbol{r}, \boldsymbol{r}_1)] \cdot [\nabla \times \boldsymbol{E}_\omega^{\mathrm{L,opt}}] - [\nabla \times G_\omega(\boldsymbol{r}, \boldsymbol{r}_1)] \cdot [\boldsymbol{n} \times \boldsymbol{E}_\omega^{\mathrm{L,opt}}] d\sigma, \tag{S6}$$

where S_R^3 is the surface of a sphere with large radius R centered at the origin. Due to the Green's function having outgoing boundary conditions (Silver-Müller radiation condition), the outgoing part of $\boldsymbol{E}_\omega^{\mathrm{L,opt}}$ does not contribute to this equation and we thus get

$$\boldsymbol{E}_\omega^{\mathrm{L,opt}}(\boldsymbol{r}_1) = \int_{S_R^3} [\boldsymbol{n} \times G_\omega(\boldsymbol{r}, \boldsymbol{r}_1)] \cdot [\nabla \times \delta \boldsymbol{E}_\omega^*(\boldsymbol{r})] - [\nabla \times G_\omega(\boldsymbol{r}, \boldsymbol{r}_1)] \cdot [\boldsymbol{n} \times \delta \boldsymbol{E}_\omega^*(\boldsymbol{r})] d\sigma. \tag{S7}$$

We can simplify this expression by using the Green's function identity

$$\begin{aligned} -2i\mathrm{Im}(G_\omega(\boldsymbol{r}_0, \boldsymbol{r}_1)) =& -2i \int_{S_R^3} (\nabla \times \mathrm{Im}(G_\omega)(\boldsymbol{r}, \boldsymbol{r}_1)) \cdot (\boldsymbol{n} \times G_\omega(\boldsymbol{r}, \boldsymbol{r}_0)) - (\boldsymbol{n} \times \mathrm{Im}(G_\omega)(\boldsymbol{r}, \boldsymbol{r}_1)) \cdot (\nabla \times G_\omega(\boldsymbol{r}, \boldsymbol{r}_0)) d\sigma \\ =& \int_{S_R^3} (\nabla \times G_\omega^*(\boldsymbol{r}, \boldsymbol{r}_1)) \cdot (\boldsymbol{n} \times G_\omega(\boldsymbol{r}, \boldsymbol{r}_0)) - (\boldsymbol{n} \times G_\omega^*(\boldsymbol{r}, \boldsymbol{r}_1)) \cdot (\nabla \times G_\omega(\boldsymbol{r}, \boldsymbol{r}_0)) d\sigma. \end{aligned} \tag{S8}$$

The second equality holds due to the outgoing boundary condition of G_ω. Note that we avoided the ambiguity of the dyadic Green's function in the source region [S1] in Eq. (S7)/(S8) because $\boldsymbol{E}_\omega^{\mathrm{L,opt}}, \mathrm{Im}(G_\omega)$ are solutions of the source free wave equation inside the system.

By combining Eq. (S5) with (S7) and (S8) this gives us the linear contribution of the field at the nonlinearities

$$\boldsymbol{E}_\omega^{\mathrm{L,opt}}(\boldsymbol{r}) = -\frac{2ik^2}{\epsilon_0} \int \mathrm{Im} G_\omega(\boldsymbol{r}, \boldsymbol{r}')(\delta \boldsymbol{P}_\omega^{\mathrm{NL}}(\boldsymbol{r}'))^* d\boldsymbol{r}'. \tag{S9}$$

Since we assumed that the probing field produces a nonlinear response, i.e., $\delta \boldsymbol{E}_\omega \neq 0$, it follows directly from the definition that $\delta \boldsymbol{P}_\omega^{\mathrm{NL}}(\boldsymbol{r},')$ must also be non-zero. Eq. (S9) therefore implies that the electric field within the nonlinear region $\boldsymbol{E}_\omega^{\mathrm{L,opt}}(\boldsymbol{r})$ can be expected to be non-zero.

Overall for general nonlinearities (e.g. multiple nonlinearities, extended nonlinearities in space) our method is able to identify incident states that focus at the spatial region, where these non-linearities are located. However, both the spatial correlations of the electromagnetic field, captured by Im $G_\omega(\boldsymbol{r}, \boldsymbol{r},')$, and the nonlinear response of the target, $\delta \boldsymbol{P}_\omega^{\mathrm{NL}}$, obfuscate the exact field intensity at the target. This highlights the need for additional system information, such as details of the nonlinearities in the system, in order to find the optimal input wavefront for focusing. In our case we overcome this by considering systems with only a single input mode that couples to the nonlinearity such as in the case of a nonlinearity made up of an antenna measuring only a singular polarization direction connected to a LNA or a point-like nonlinearity in restricted systems with only a singular polarization degree of freedom.

C. Point-like nonlinearity

We consider a point-like nonlinearity in the Rayleigh regime with center at $\boldsymbol{r}_0$ volume V_R and diameter R, so that the far field is given by

$$\boldsymbol{E}_\omega(\boldsymbol{r}) = \boldsymbol{E}_\omega^{\mathrm{L}}(\boldsymbol{r}) + \frac{k^2 V_R}{\epsilon_0} G_\omega(\boldsymbol{r}, \boldsymbol{r}_0) \boldsymbol{P}_\omega^{\mathrm{NL}}[\boldsymbol{E}(\boldsymbol{r}_0)]. \tag{S10}$$

Thus the outgoing field pattern is only given by the Green's tensor of the linear system $G_\omega(\boldsymbol{r}, \boldsymbol{r}_0)$ coupling the location of the nonlinearity to the far field. Based on this we can simplify Eq. (S9) giving us the linear contribution

$$\boldsymbol{E}_\omega^{\mathrm{L,opt}}(\boldsymbol{r}_0) = -\mathrm{Im} G_\omega(\boldsymbol{r}_0, \boldsymbol{r}_0) \frac{2ik^2 V_R}{\epsilon_0} (\delta \boldsymbol{P}_\omega^{\mathrm{NL}}(\boldsymbol{r}_0))^*. \tag{S11}$$

Thus in the linear reference system a focus can be created at the location of the nonlinearity. In the case, where $\boldsymbol{P}_\omega^{\mathrm{NL}}$ is restricted to a singular polarization direction (e.g. antenna) then only a singular input wavefront can focus onto our target and the focus is therefore optimal.

II. FAR FIELD DESCRIPTION

We consider the case of waves at the frequency ω and a basis of the wavefronts in the far field $\boldsymbol{E}_\omega^n$ so that the in- and outgoing fields have the corresponding coefficients $\boldsymbol{c}^{\text{in}}, \boldsymbol{c}^{\text{out}}$ (see section II A).We will for now focus on the case, where the nonlinearity only couples to the system through a single polarization degree of freedom (e.g. an antenna or a system allowing only a single polarization direction), while the more general case is considered in section II E. In linear systems these coefficients can be connected by the linear scattering matrix S^{L}. However in our case this does no longer hold true due to the nonlinearity violating the superposition principle, where the connection $\boldsymbol{c}^{\text{out}} = \hat{S}(\boldsymbol{c}^{\text{in}})$ is given by the non-linear scattering operator $\hat{S}$. Luckily, those incident fields that do not interact with the nonlinearity stay linear, which means that for a point-like nonlinearity an incident field quantified by $\boldsymbol{d}^*$ can be found, so that all orthogonal incident fields do not interact with the nonlinearity, i.e.

$$\hat{S}(\boldsymbol{c}^{\text{in}}) - S^{\text{L}}\boldsymbol{c}^{\text{in}} = 0 \tag{S12}$$

whereby S^{L} is the scattering matrix of the linear system without the nonlinearity. This can be seen by considering that the vector $\boldsymbol{d}$ corresponds to the far field coefficients of $G_\omega(\boldsymbol{r}, \boldsymbol{r}_0)$ in our chosen basis, i.e. $d_j = \frac{-i}{2\omega\mu_0}\int_{S_R^3}(\boldsymbol{n} \times (\boldsymbol{E}_\omega^j)(r)) \cdot (\nabla \times G_\omega(r, r_0)) - (\nabla \times (\boldsymbol{E}_\omega^j)(r)) \cdot (\boldsymbol{n} \times G_\omega(r, r_0)) d\sigma$ (see section II A). Using Green's identity we can see that $d_j = \frac{i}{2\omega\mu_0}(\boldsymbol{E}_\omega^{\text{L},j})(r_0)$, where $\boldsymbol{E}_\omega^{\text{L},j}$ corresponds to the solution of the linear system of the far field mode j. Thus if we take an incident field in the input channels $\boldsymbol{c}^{\text{in}} \cdot \boldsymbol{d} = 0$ then the linear part of the field is given by $\boldsymbol{E}_\omega^{\text{L}}(\boldsymbol{r}_0) = \sum_j c_j \boldsymbol{E}_\omega^{\text{L},j}(\boldsymbol{r}_0) \propto \boldsymbol{d} \cdot \boldsymbol{c} = 0$, showing that no field is present at the location of the nonlinearity. Due to the non-linear polarization P^{NL} being only non-zero for electromagnetic fields that interact with the non-linearity ($E(r_0) \neq 0$), this means that only the components of $\boldsymbol{c}^{\text{in}}$ parallel to $\boldsymbol{d}^*$ result in differences between $\hat{S}$ and S^{L}, i.e.

$$\hat{S}(\boldsymbol{c}^{\text{in}}) - S^{\text{L}}\boldsymbol{c}^{\text{in}} = \hat{S}(\hat{P}_{\boldsymbol{d}^*}\boldsymbol{c}^{\text{in}}) - S^{\text{L}}\hat{P}_{\boldsymbol{d}^*}\boldsymbol{c}^{\text{in}}, \tag{S13}$$

where $\hat{P}_{\boldsymbol{d}^*} = \boldsymbol{d}^*\boldsymbol{d}^T/|\boldsymbol{d}|^2$ is the projection operator onto the vector $\boldsymbol{d}^*$. We can simplify this even further by noting that Eq. (S10) tells us that the part of the outgoing field due to the non-linearity results in an outgoing field given by $G_\omega(\boldsymbol{r}, \boldsymbol{r}_0)$, which is described by $\boldsymbol{d}$ so that we get

$$\hat{S}(\boldsymbol{c}^{\text{in}}) - S^{\text{L}}\boldsymbol{c}^{\text{in}} = \hat{P}_{\boldsymbol{d}}\left(\hat{S}(\hat{P}_{\boldsymbol{d}^*}\boldsymbol{c}^{\text{in}}) - S^{\text{L}}\hat{P}_{\boldsymbol{d}^*}\boldsymbol{c}^{\text{in}}\right), \tag{S14}$$

where $\hat{P}_{\boldsymbol{d}}$ is the projection operator onto the vector $\boldsymbol{d}$. By inserting the definition of the orthogonal projection operators we can now describe the scattering of the nonlinear scattering operator by

$$\hat{S}(\boldsymbol{c}^{\text{in}}) = S^{\text{L}}\boldsymbol{c}^{\text{in}} + \frac{\boldsymbol{d}\boldsymbol{d}^\dagger}{|\boldsymbol{d}|^2}\left(\hat{S}\left(\frac{\boldsymbol{d}^*\boldsymbol{d}^T\boldsymbol{c}^{\text{in}}}{|\boldsymbol{d}|^2}\right) - S^{\text{L}}\frac{\boldsymbol{d}^*\boldsymbol{d}^T\boldsymbol{c}^{\text{in}}}{|\boldsymbol{d}|^2}\right) = S^{\text{L}}\boldsymbol{c}^{\text{in}} + \boldsymbol{d}f(\boldsymbol{d}\cdot\boldsymbol{c}^{\text{in}}), \tag{S15}$$

whereby S^{L} is the scattering matrix of the linear system without the nonlinearity and f is the nonlinear scalar function describing the deviation of $\hat{S}$ from a linear relation, i.e. $f(x) = \boldsymbol{d}^\dagger[\hat{S}(\boldsymbol{d}^*x/|\boldsymbol{d}|^2) - S^L\boldsymbol{d}^*x/|\boldsymbol{d}|^2]/|\boldsymbol{d}|^2$. This nonlinear operator can now be probed using a basis of incident channels, which can be conveniently summarized in the matrix S^α so that we have

$$S^\alpha_{m,n} = S^{\text{L}}_{m,n} + d_m\alpha^{-1}f(\alpha d_n), \tag{S16}$$

where we choose the basis of incident channels $\boldsymbol{c}^{\text{in}} = \alpha\boldsymbol{e}_n$ for all n, quantify the strength of the incident fields by α and normalize the outgoing fields by α. This has the advantage that for linear system this reduces to the scattering matrix, while we can probe the nonlinear contributions of the scattering operator by varying α. Note however that S^α does not fully describe the operator $\hat{S}$, due to the loss of the superposition principle in nonlinear systems, but only the response of the system for a basis of incident channels.

A. Connecting the far field patterns with the scattering matrix

We connect the scattering matrix with the far field pattern using the hermitian form based on the poynting vector [S2]

$$\langle E_\omega^1, E_\omega^2\rangle = \frac{1}{2}\int_{S_R^3}[(E_\omega^1)^* \times H_\omega^2 - (H_\omega^1)^* \times E_\omega^2] \cdot d\boldsymbol{n}, \tag{S17}$$

where S_R^3 is the surface of a sphere with large radius R centered at the origin. We can now use this product to define a basis of incident states $\boldsymbol{E}_\omega^j$ with power normalization, i.e.

$$\langle E_\omega^i, E_\omega^j \rangle = -\delta_{i,j}. \tag{S18}$$

We now define the input coefficients c^{in} corresponding to a field E_ω by

$$c_i^{\text{in}} = -\langle E_\omega^i, E_\omega \rangle. \tag{S19}$$

Similarly the outgoing field if given by $(E_\omega^j)^*$, which can be used to define the outgoing coefficients c^{out} by

$$c_i^{\text{out}} = \langle (E_\omega^i)^*, E_\omega \rangle. \tag{S20}$$

The scattering operator is now given by the relation $c^{\text{out}} = \hat{S}[c^{\text{in}}]$, which is linear in the case where no non-linearity is present. Finally, we can define the outgoing coefficients $\boldsymbol{d}$ corresponding to the Green's function (e.g. $E_\omega(r) = G_\omega(r, r_0)$, $H_\omega = \nabla \times E_\omega/(i\mu_0\omega)$) by

$$d_i = \langle (E_\omega^i)^*, G_\omega(\cdot, r_0) \rangle = \frac{-i}{2\omega\mu_0} \int_{S_R^3} (\boldsymbol{n} \times (\boldsymbol{E}_\omega^j)(r)) \cdot (\nabla \times G_\omega(r, r_0)) - (\nabla \times (\boldsymbol{E}_\omega^j)(r)) \cdot (\boldsymbol{n} \times G_\omega(r, r_0)) d\sigma. \tag{S21}$$

B. Extracting the focusing field

We will now use Eq. (S16) at two incident powers $\alpha, \alpha + \Delta\alpha$ to define a matrix version of Eq. (S5). The results are summarized by the matrices $S^\alpha, S^{\alpha+\Delta\alpha}$, which gives us

$$\Delta S_{m,n} = (S^{\alpha+\Delta\alpha} - S^\alpha)_{m,n} = d_m \left((\alpha + \Delta\alpha)^{-1} f((\alpha + \Delta\alpha) d_n) - \alpha^{-1} f(\alpha d_n) \right). \tag{S22}$$

This rank-one matrix contains information of at least one incident channel that interacts with the nonlinearity, allowing us to apply a singular value decomposition on ΔS to extract $\boldsymbol{d}$. Note that if multiple far field modes couple to the nonlinearity (e.g. non-scalar waves, multiple polarization degrees of freedom) or if multiple nonlinearities are present then the non-linearity acts on the subspace of these modes. In this case f would need to be replaced by a function describing the nonlinear interaction between these modes and the rank of the matrix ΔS will be the dimension of this subspace (see section II E). In general while we can still use ΔS to identify modes that focus on the nonlinearity, in order to create an optimal focus more information on the non-linearity is needed.

C. Reciprocity and nonlinearity detection

One important property in most linear systems is the reciprocity condition, which can now break due to nonlinear interactions. In our case this turns out to be

$$(S^\alpha - (S^\alpha)^T)_{m,n} = d_m \alpha^{-1} f(\alpha d_n) - d_n \alpha^{-1} f(\alpha d_m). \tag{S23}$$

While in theory it is not guaranteed that we will see reciprocity breaking using S^α (e.g. if $\boldsymbol{d}$ corresponds to a basis vector $d_n \propto \delta_{m,n}$), in practice we saw that this can serve as a useful tool for the detection of the nonlinearity.

D. Lowest singular value and absorption

We find a strong average correlation between the wavefront for maximal focusing $\boldsymbol{c}^{\text{in}} = \boldsymbol{d}^*$ and the eigenvector $\boldsymbol{U}_N$ corresponding to the smallest eigenvalue σ_N of $(S^\alpha)^\dagger S^\alpha$ that gives minimal reflection from the cavity and therefore maximal absorption [S3]. The correlation coefficient averaged over the frequency range is $\langle |\boldsymbol{U}_N^* \cdot \boldsymbol{c}^{\text{in}}| \rangle \sim 0.8$. As the cavity is closed, absorption at the target is indeed the main loss mechanism. The correlation coefficient is however below unity since other dissipative mechanisms such as uniform absorption within the cavity also takes place. Small eigenvalues $\sigma_N \to 0$ indicate that the incident energy is almost completely dissipated within the target.

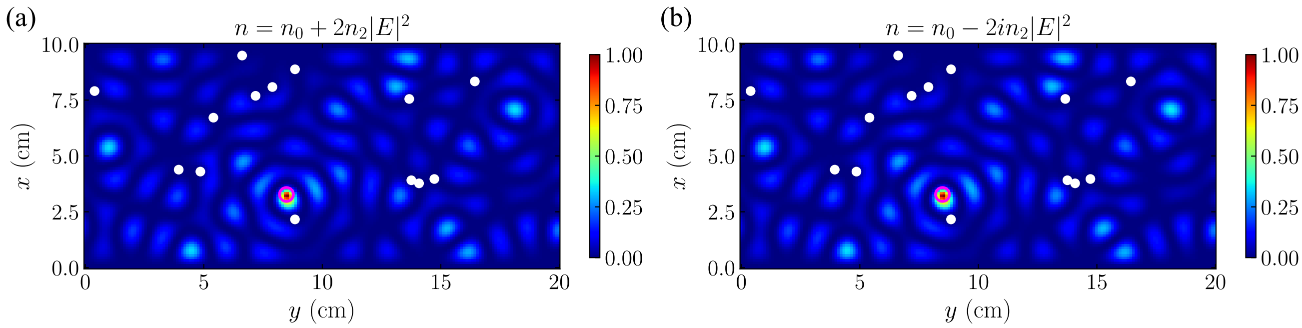

FIG. S1. (a, b) Intensity map within the system for $\boldsymbol{c}^{\text{in}} = \boldsymbol{d}^*$ for a nonlinearity which intensity-dependent index is given by Eq. (S26). Panel (a) corresponds to case 1 and panel (b) to case 2. Both maps are normalized by the maximum value obtained for the optimized wavefront. The white dots represent the location of the metallic scatterers and the pink circle the location of the nonlinear target.

E. Extended and multiple nonlinearities

We will now consider extended and multiple nonlinearities D_i. Similar to section II the far field modes that couple to the regions of these nonlinearities are spanned by $G_\omega(\boldsymbol{r}, \boldsymbol{r}_0)$ for $\boldsymbol{r}_0 \text{in} D = \cup_i D_i$, which we will describe in the far field by the basis $\boldsymbol{d}^1, \boldsymbol{d}^2, \ldots, \boldsymbol{d}^N$ of dimension N. With the same arguments as in section II we can now show that for an incident field quantified by $\boldsymbol{c}$ and $\boldsymbol{c} \cdot \boldsymbol{d}^i = 0$ for all i, the corresponding electric field in the linear system disappears at the locations of the nonlinearities, i.e. $E^L_\omega(r_0) = 0$ for $\boldsymbol{r}_0 \text{in} D$. As such the vectors $(\boldsymbol{d}^i)^*$ span the space of input modes that can couple to the nonlinearity and $\boldsymbol{d}^i$ the outgoing modes that the nonlinearity couples to. Using this we now write the scattering operator

$$\hat{S}[\boldsymbol{c}^{\text{in}}] = S^{\text{L}} \boldsymbol{c}^{\text{in}} + \sum_{i=1}^{N} \boldsymbol{d}^i f_i(\boldsymbol{d}^1 \cdot \boldsymbol{c}^{\text{in}}, \ldots, \boldsymbol{d}^N \cdot \boldsymbol{c}^{\text{in}}), \tag{S24}$$

for the non-linear functions f_i. Next we consider the difference matrix given by

$$\Delta S_{m,n} = \sum_i d^i_m \left((\alpha + \Delta\alpha)^{-1} f_i((\alpha + \Delta\alpha) d^1_n, \ldots, (\alpha + \Delta\alpha) d^N_n) - \alpha^{-1} f_i(\alpha d^1_n, \ldots, \alpha d^N_n) \right). \tag{S25}$$

We can see that the dimension of ΔS is at most the number of in coupling modes N and that the space of left singular values is spanned by $\boldsymbol{d}^i$. This shows that we can use ΔS to extract the incident wavefront that focus on the nonlinearities. However it is important to note that in general the vectors $\boldsymbol{d}^i$ will not be the set of left singular vectors, due to $\left((\alpha + \Delta\alpha)^{-1} f_i((\alpha + \Delta\alpha) d^1_n, \ldots, (\alpha + \Delta\alpha) d^N_n) - \alpha^{-1} f_i(\alpha d^1_n, \ldots, \alpha d^N_n) \right)$ being in general not orthogonal for different i.

III. SIMULATIONS FOR CONTROLLED NONLINEAR BEHAVIOR

We present the results of numerical simulations obtained with COMSOL. The goal here is to demonstrate the versatility of the method with respect to the type of nonlinearity present in the system. We consider a 2D cavity with 9 antennas on each side at a frequency $f = 13.51$ GHz. The field inside the cavity is randomized through the presence of 14 randomly located metallic scatterers of 1 mm radius. The nonlinear target is defined as a dielectric scatterer with a 1 mm radius and Kerr-like index given by:

$$n = \begin{cases} n_0 + n_2 |\boldsymbol{E}_\omega|^2 & \text{case 1} \\ n_0 - i n_2 |\boldsymbol{E}_\omega|^2 & \text{case 2,} \end{cases} \tag{S26}$$

where the case 1 corresponds to nonlinear dispersion, and the case 2 corresponds to nonlinear absorption. The intensity maps obtained from the first left singular vectors of ΔS in each cases (as described in the main text) are shown in Fig. S1. The same locations are used for the scatterers and the target inside the system, and for the value of the nonlinear response $n_2 = 5 \cdot 10^{-6}$ $\mathrm{m}^2/\mathrm{V}^2$. In both cases the incident wavefront obtained from the SVD focuses on the target regardless of the chosen nonlinear function, showing that the approach is independent of the type of nonlinearity.

IV. CORRECTION OF THE PHASE FOR BROADBAND SIGNALS

The wavefront for optimal focusing in space and time corresponds to the time-reversed (or equivalently phase-conjugate) of the transmission coefficient $\boldsymbol{c}^{\mathrm{opt}}(\omega) = \boldsymbol{t}^*(\omega)$. The phases at each frequency are aligned at the focus, meaning that the scattered wavefront $\boldsymbol{c}^{\mathrm{opt}}(\omega) = \Delta S(\omega)\boldsymbol{c}^{\mathrm{opt}}(\omega)$ acquires a phase equal to $\arg[t(\omega)]$. We therefore determine the phase $\phi(\omega)$ from the condition $\arg\left[\boldsymbol{c}^{\mathrm{in}} e^{i\phi} \Delta S \boldsymbol{c}^{\mathrm{in}} e^{i\phi}\right] = 0$. Because both $\boldsymbol{c}^{\mathrm{in}} e^{i\phi}$ and $\boldsymbol{c}^{\mathrm{in}} e^{i(\phi+\pi)}$ satisfy this condition, we finally exploit the continuity of $\phi(\omega)$ over the bandwidth to correct π-phase shifts.

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